Elementary Theory of Sets

Chapter 1: Sets and Basic Operations on Sets

Teaching Note for Students

Part I — Elementary Theory of Sets

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1 Introduction

The concept of a **set** is one of the most fundamental ideas in all of mathematics. It formalises the intuitive notion of *grouping objects together* and treating them as a single entity.

Three basic operations on sets: Union $\cup$ Intersection $\cap$ Complement ^{c}

Set theory is deeply connected to **logic**: Venn diagrams — pictorial representations of sets — can be used to verify logical arguments and determine the validity of statements. This connection will be explored fully when we study Boolean algebra.

2 Sets and Elements

Definition: Set

A **set** is any well-defined collection of objects. The objects are called the **elements** (or **members**) of the set.

2.1 Notation

- Sets are denoted by **capital letters**: $A, B, X, Y, \ldots$
- Elements are denoted by **lower-case letters**: $a, b, c, x, y, z, \ldots$
- $p \in A$ means "p is an element of A".
- $p \notin A$ means "p is *not* an element of A".

2.2 Two Ways to Specify a Set

1. Tabular (Roster) Form

List all elements inside braces, separated by commas.

Example:

$$A = \{\text{a, e, i, o, u}\}$$

(A is the set of English vowels.)

2. Set-Builder (Property) Form

Describe the *property* that characterises members.

Example:

$$B = \{x : x \text{ is an even integer, } x > 0\}$$

("B is the set of x such that x is a positive even integer.")

Note

A set remains the **same** even if its elements are repeated or rearranged. For instance, $\{1, 2, 1, 6/3\} = \{1, 2\}$.

2.3 Standard Number Sets

Symbol	Name	Description
$\mathbb{N}$	Natural / Non-negative integers	$0, 1, 2, 3, \ldots$
$\mathbb{P}$	Positive integers	$1, 2, 3, \ldots$
$\mathbb{Z}$	Integers	$\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	Rational numbers	fractions p/q, $q \neq 0$
$\mathbb{R}$	Real numbers	all points on the number line
$\mathbb{C}$	Complex numbers	$a + bi,\ \ i = \sqrt{-1}$

Example 1.1

(a) $A = \{\mathrm{a, e, i, o, u}\}$ can also be written as $A = \{x : x \text{ is a vowel in the English alphabet}\}$. Note $b \notin A$ but $e \in A$.

(b) $B = \{2, 4, 6, \ldots\}$: we have $8 \in B$ but $9 \notin B$.

(c) $E = \{x : x^2 - 3x + 2 = 0\} = \{1, 2\}$ — the solution set of the equation.

3 Universal Set and Empty Set

Definition: Universal Set

The **universal set U** is the large fixed set that contains all sets under consideration in a given context.

- In plane geometry: U = all points in the plane.
- In population studies: U = all people in the world.

Definition: Empty Set (Null Set)

The **empty set** (or **null set**), denoted $\varnothing$, is the set that contains *no* elements.

Example: $S = \{x : x \text{ is a positive integer},\ x^2 = 3\} = \varnothing$ (no positive integer satisfies $x^2 = 3$).

There is only one empty set.

4 Subsets

Definition: Subset

A is a **subset** of B, written $A \subseteq B$ (or $B \supseteq A$), if every element of A is also an element of B.

$$A \subseteq B \iff [\, x \in A \Rightarrow x \in B \,]$$

If A is *not* a subset of B we write $A \not\subseteq B$.

4.1 Key Properties of Subsets

Theorem: 1.1 — Subset Properties

For any sets A, B, C:

(i) $\varnothing \subseteq A \subseteq U$

(ii) $A \subseteq A$ (reflexivity)

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (transitivity)

(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

4.2 Proper Subset and Disjoint Sets

Definition: Proper Subset

A is a **proper subset** of B, written $A \subset B$, if $A \subseteq B$ and $A \neq B$ (i.e. B has at least one element not in A).

Definition: Disjoint Sets

Sets A and B are **disjoint** if they share no common elements, i.e. $A \cap B = \varnothing$.

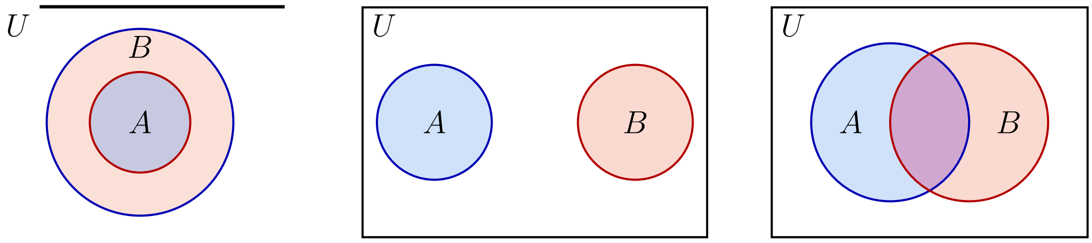
Example 1.2

Let $A = \{1, 3, 5, 8, 9\}$, $B = \{1, 2, 3, 5, 7\}$, $C = \{1, 5\}$.

- $C \subseteq A$ and $C \subseteq B$ since 1 and 5 belong to both A and B.
- $B \not\subseteq A$ because $2 \in B$ but $2 \notin A$.
- $\mathbb{P} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

5 Venn Diagrams

A **Venn diagram** is a pictorial representation of sets using enclosed regions in the plane. The universal set U is a rectangle; each set is a disk inside it.



(a) $A \subseteq B$ (b) A and B are disjoint (c) General case

Figure 1: Venn diagrams illustrating set relationships

6 Set Operations

6.1 Union

Definition: Union

The **union** of sets A and B, denoted $A \cup B$, is the set of all elements belonging to A *or* B (or both):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

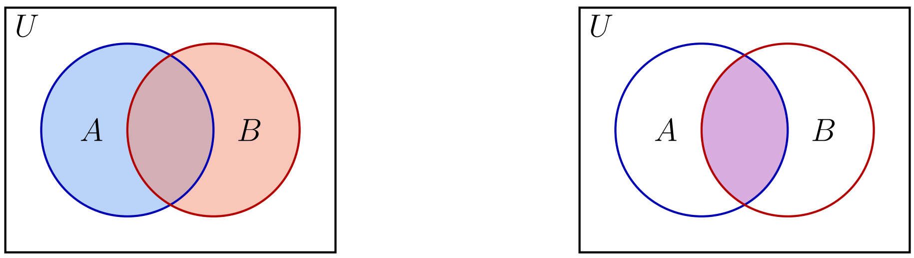
6.2 Intersection

Definition: Intersection

The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements belonging to *both* A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \varnothing$, then A and B are **disjoint**.



(a) $A \cup B$ is shaded (b) $A \cap B$ is shaded

Figure 2: Union and Intersection

Theorem: 1.2 — Union and Intersection Inequalities

For any sets A and B:

$$A \cap B \subseteq A \subseteq A \cup B \qquad \text{and} \qquad A \cap B \subseteq B \subseteq A \cup B$$

Example 1.3

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$.

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}, \qquad A \cap B = \{3, 4\}$$
$$A \cup C = \{1, 2, 3, 4, 8, 9\}, \qquad A \cap C = \{2, 3\}$$
$$B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}, \qquad B \cap C = \{3\}$$

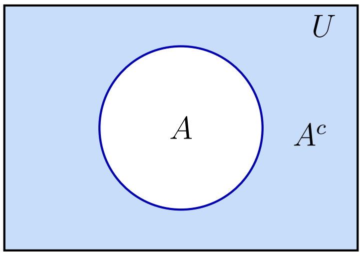
6.3 Complement

Definition: Complement

The **absolute complement** (or simply **complement**) of a set A, denoted A^c, is the set of all elements in U that are *not* in A:

$$A^c = \{x : x \in U,\ x \notin A\}$$

Other notations: A' or $\bar{A}$.



A^c is shaded

Figure 3: The complement A^c

Example 1.4

Let $U = \{a, b, c, \dots, y, z\}$ (the English alphabet), $A = \{a, b, c, d, e\}$, and $V = \{a, e, i, o, u\}$ (vowels). Then

$$A^c = \{f, g, h, \dots, z\}, \qquad V^c = \text{all consonants.}$$

6.4 Difference and Symmetric Difference

Definition: Set Difference

The **difference** (or relative complement) of A and B, written $A \setminus B$ (also $A - B$), is:

$$A \setminus B = \{x : x \in A,\ x \notin B\}$$

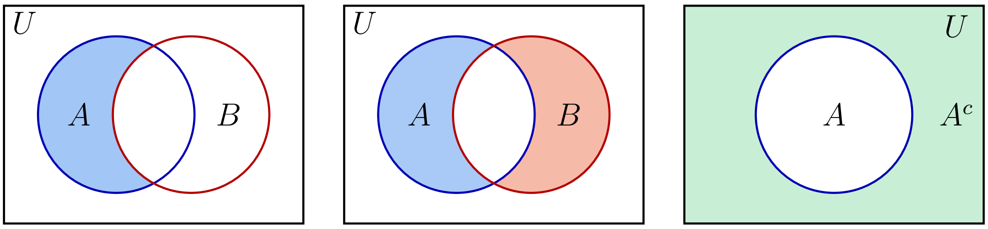
Read as "A minus B".

Definition: Symmetric Difference

The **symmetric difference** of A and B, written $A \oplus B$, is:

$$A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

It consists of elements in exactly *one* of A or B, but not both.



(a) $A \setminus B$ shaded (b) $A \oplus B$ shaded (c) A^c shaded

Figure 4: Set difference, symmetric difference, and complement

Example 1.5

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{6, 7, 8, 9\}$.

$$A \setminus B = \{1, 2\}, \qquad B \setminus A = \{5, 6, 7\}$$
$$A \oplus B = \{1, 2, 5, 6, 7\}, \qquad B \oplus C = \{3, 4, 5, 8, 9\}$$

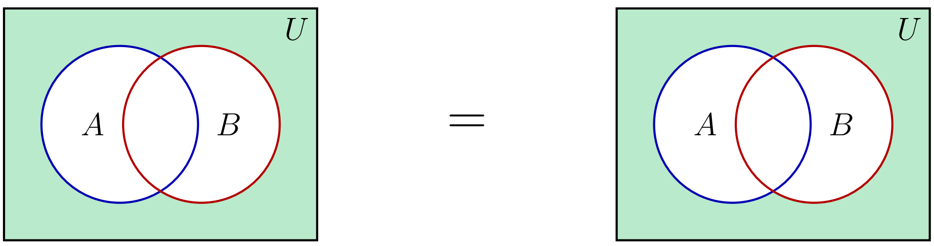
Note that A and C are disjoint, so $A \setminus C = A$, $C \setminus A = C$, and $A \oplus C = A \cup C$.

7 Algebra of Sets and Duality

Theorem: 1.4 — Laws of the Algebra of Sets

	Idempotent Laws		
(1a)	$A \cup A = A$	(1b)	$A \cap A = A$
	Associative Laws		
(2a)	$(A \cup B) \cup C = A \cup (B \cup C)$	(2b)	$(A \cap B) \cap C = A \cap (B \cap C)$
	Commutative Laws		
(3a)	$A \cup B = B \cup A$	(3b)	$A \cap B = B \cap A$
	Distributive Laws		
(4a)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	Identity Laws		
(5a)	$A \cup \varnothing = A$	(5b)	$A \cap U = A$
(6a)	$A \cup U = U$	(6b)	$A \cap \varnothing = \varnothing$
	Involution Law		
	(7) $(A^c)^c = A$		
	Complement Laws		
(8a)	$A \cup A^c = U$	(8b)	$A \cap A^c = \varnothing$
(9a)	$U^c = \varnothing$	(9b)	$\varnothing^c = U$
	De Morgan's Laws		
(10a)	$(A \cup B)^c = A^c \cap B^c$	(10b)	$(A \cap B)^c = A^c \cup B^c$

7.1 De Morgan's Laws — Illustrated



$(A \cup B)^c$ shaded $\qquad\qquad$ $A^c \cap B^c$ shaded

Figure 5: De Morgan's Law (10a): $(A \cup B)^c = A^c \cap B^c$

7.2 Principle of Duality

Principle of Duality

The **dual** E^* of any equation E in set algebra is obtained by replacing:

$$\cup \leftrightarrow \cap \qquad U \leftrightarrow \varnothing$$

If E is an identity, then its dual E^* is also an identity.

Example: $(\mathbf{U} \cap A) \cup (B \cap A) = A$ is dual to $(\varnothing \cup A) \cap (B \cup A) = A$.

8 Finite Sets and Counting Principles

Definition: Finite Set

A set A is **finite** if it contains exactly m distinct elements for some nonnegative integer m. We write $n(A)$ or $|A|$ for this count. Otherwise A is **infinite**.

Theorem: Inclusion-Exclusion Principle

For any two finite sets A and B:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

When A and B are **disjoint**: $n(A \cup B) = n(A) + n(B)$.

For three sets:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

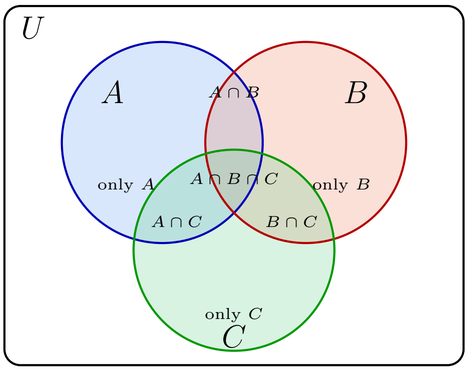


Figure 6: Three-set Venn diagram with all regions labelled

Example Counting

In a class of 50 students: 30 study Mathematics (M), 22 study Physics (P), and 10 study both.

$$n(M \cup P) = 30 + 22 - 10 = 42.$$

So $50 - 42 = 8$ students study neither subject.

9 Summary of Key Symbols

Symbol	Meaning	Example
$\in$	is an element of	$3 \in \{1, 2, 3\}$
$\notin$	is not an element of	$4 \notin \{1, 2, 3\}$
$\subseteq$	is a subset of	$\{1, 2\} \subseteq \{1, 2, 3\}$
$\subset$	is a proper subset of	$\{1, 2\} \subset \{1, 2, 3\}$
$=$	set equality	$\{1, 2\} = \{2, 1\}$
$\varnothing$	empty (null) set	$\{x : x^2 = -1, x \in \mathbb{R}\} = \varnothing$
U	universal set	contains all sets in context
A^c	complement of A	$U \setminus A$
$A \cup B$	union	all elements in A or B
$A \cap B$	intersection	elements in both A and B
$A \setminus B$	difference (A minus B)	elements in A not in B
$A \oplus B$	symmetric difference	elements in exactly one of A, B
$n(A)$	cardinality (size) of A	$n(\{a, b, c\}) = 3$

10 Practice Problems

Q1. Write the following sets in tabular form:

(a) $A = \{x \in \mathbb{Z} : -3 < x \leq 2\}$

(b) $B = \{x : x^2 - 5x + 6 = 0\}$

(c) $C = \{x \in \mathbb{N} : x \text{ is a factor of } 12\}$

Q2. Let $U = \{1, 2, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{5, 6, 7, 8, 9\}$. Find: (a) $A \cup B$, (b) $A \cap C$, (c) A^c, (d) $B \setminus C$, (e) $A \oplus B$, (f) $(A \cup B) \cap C^c$.

Q3. Verify De Morgan's Law $(A \cup B)^c = A^c \cap B^c$ using U, A, B from Q2 above.

Q4. Draw a Venn diagram and shade the region representing $(A \cap B^c) \cup (A^c \cap B)$. What standard operation does this equal?

Q5. In a survey of 100 people: 65 like tea (T), 40 like coffee (C), and 20 like both.

(a) Find $n(T \cup C)$.

(b) How many people like neither tea nor coffee?

(c) How many like only tea?

Q6. Prove that for any sets A and B: $A \subseteq B \iff A \cap B = A \iff A \cup B = B$.

Q7. Show by Venn diagram and algebraically that: $A \setminus B = A \cap B^c$.

Q8. If $A \subseteq B$, simplify: (a) $A \cup B$, (b) $A \cap B$, (c) $A \setminus B$, (d) $B \setminus A$.

Remember: Understanding *why* these laws work through Venn diagrams and logical reasoning is more valuable than memorising formulas. Always verify algebraic identities with concrete examples first!